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DIFFRACTION BY A DIELECTRIC WEDGE
WITH APPLICATION TO PROPAGATION THROUGH A COLD FRONT

by

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1. Introduction

The problem of the diffraction of plane waves by a dielectric wedge is of considerable physical and mathematical interest. It corresponds to the passage of light thru a prism, or the diffraction by a cold front above a plane earth.

The diffraction of plane or spherical electromagnetic waves by a perfectly conducting wedge was studied by H. M. Macdonald (1), who used the method of expansion in eigenfunctions in cylindrical coordinates. More recently, the diffraction of pulses by perfectly conducting wedges has been treated (2), using the method of conical flows.

Although the geometry of the dielectric wedge is the same as that of the conducting wedge, the fact that the dielectric "constant" is a piecewise constant function of angle means that the differential equations and boundary conditions of the system are not separable in polar coordinates. Hence, methods involving separation in polar coordinates, such as the use of a transform theorem involving Bessel functions integrated over their order(3), which appears superficially promising because the boundary conditions on the wedge are independent of the radius, may be expected to fail.

In view of this circumstance, an approximation method has been developed. If the dielectric constant within the wedge does not differ too greatly from that outside it, an expansion of the solution in powers of the percentage difference in dielectric constant should converge fairly rapidly, and so a few terms of the series should give a good approximation to the entire solution. In particular, we will restrict ourselves to considering

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(1) H. M. Macdonald, "Electromagnetism", G. Bell 1934, pp. 78 ff.

(2) J. B. Keller & A. Blank, "Diffraction & Reflection of Pulses by Wedges and Corners", N.Y.U. Math. Research Group Report No. EM-21

(3) N. N. Lebedev, "Doklady Akad. Nauk. SSSR 58, 1007 (1947)

the first two terms of the series. The zero order term represents the incident field, and the first order term represents the diffraction effects. Since for a cold front the percentage difference in dielectric constant actually is small, we will restrict ourselves to considering this application, although the method used is not so restricted. Hence, we place the wedge over a perfectly conducting earth.

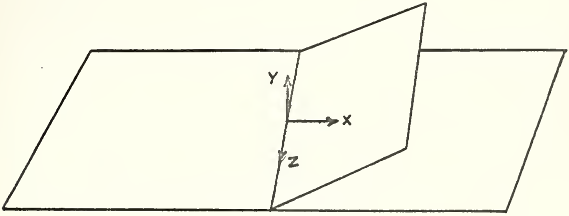
In Section II, we use Snell's laws to obtain a "geometrical" solution. In Sections III-VI, the actual solution is found, whose plane-wave part agrees with the solution obtained in Section II. The diffracted part of the field is expressed in the form of an outgoing cylindrical wave whose angular pattern is determined. Finally, the details of the application to the cold front are considered in Section VII.

2. The Geometrical Solution

We shall first discuss the plane wave or geometrical part of the solution, which may be obtained by using Snell's laws and neglecting the diffracted waves.

We must first distinguish between the possible polarizations. Let us introduce a system of coordinates as in Figure 1, with the x-axis horizontal, the y-axis vertical, and the z-axis parallel to the interface. Thus:

Figure 1



Now we wish to treat only the case where the waves are incident normally on the wedge. Thus the direction of propogation of the incident wave is normal to the z-axis. If the incident wave is a uniform plane wave, so that it has no variations of phase or amplitude in the z-direction, the total field will also have no variation in the z-direction. Hence, we reduce to a two-dimensional situation.

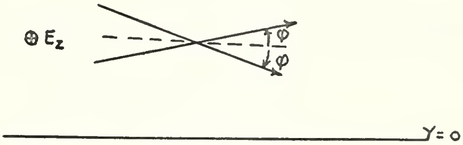
We may now further separate the problem into two cases. Any two-dimensional electromagnetic field may be separated into the sum of two parts in one of which E is entirely in the z-direction, and in the other H is entirely in the z-direction. These will not couple to each other, since the Maxwell equations break up into two groups. We may accordingly consider them separately. Since the behavior of the two situations does not differ significantly, we shall only consider the case where E is in the z-direction, so $E = (0,0,E_z)$.

In this case, since we choose the ground to be perfectly conducting, and E_z is tangential to the ground, E_z must vanish for $y = 0$. Accordingly, we describe a field traveling over the ground not by one but by two waves of equal and opposite amplitude, with equal horizonal propagation constants, and equal and opposite vertical propagation constants, so:

$$(2,1) \quad E_z = E_0 \exp(ik\, x \cos\varphi) \left[\exp(iky \sin\varphi) - \exp(-iky \sin\varphi)\right] \exp(-i\omega t)$$

Here E_0 is the amplitude, $k = \omega/c = 2\pi/\lambda$, and φ is the angle from the horizontal. Graphically:

Figure 2



Here we have chosen the angle φ to represent propagation to the right. If $\pi/2 < \varphi < \pi$, the wave will propagate to the left. The two cases must be considered separately. We now wish to have this field configuration impinge upon the wedge.

When a wave strikes the wedge, it is partly reflected and partly refracted. Let us first consider incident waves traveling to the right. The reflected wave will go off at a direction making equal angles with the normal to the wedge. The refracted wave will be bent towards the normal, since the wedge medium has a higher index of refraction than the atmosphere. If the direction of incidence and index of refraction are suitably related, the wave will be bent so that it is traveling downward. It will then strike the earth and be reflected. Thus we have Figure 3a, 3b:

Fig. 3a

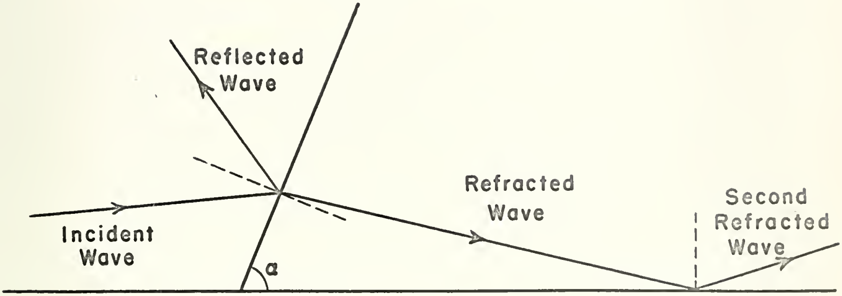
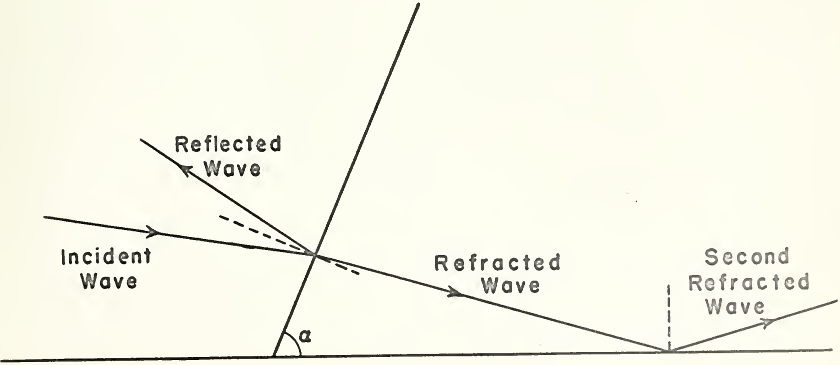

Reflected Wave
Refracted Wave
Second Refracted Wave
Incident Wave
a

Fig. 3b.

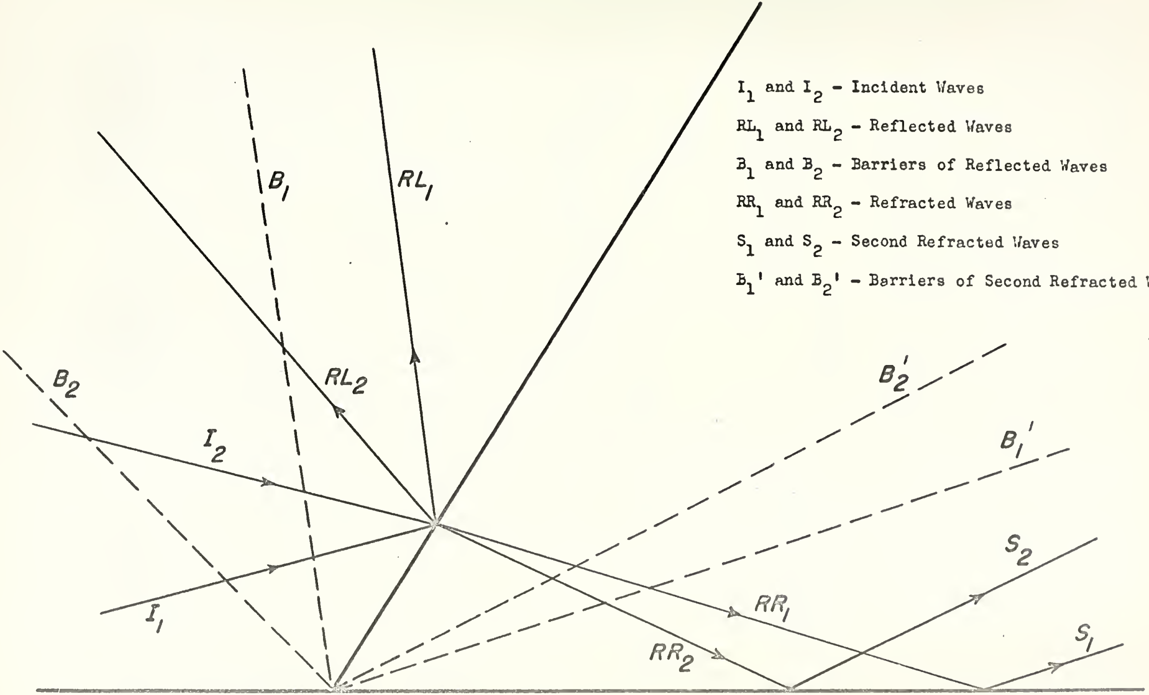

Reflected Wave
Incident Wave
Refracted Wave
Second Refracted Wave
a

The direction of the refracted wave makes a smaller angle with the ground than does the normal to the interface, whence it immediately follows that the second refracted wave will not strike the interface, and no other waves will be generated.

The incident wave is present in the entire region to the left of the wedge. This is not true of the reflected wave. Since the reflected wave must be generated by waves striking the interface, it will not appear in regions to which such waves cannot penetrate. Thus, draw a line parallel to the direction of the reflected wave and passing through the origin. The reflected wave will then only exist in the region between this line and the interface. This is because the lowest point on the interface, that is, the origin, gives rise to a reflected wave along this dividing line, and no waves can appear beyond. By identical reasoning, the second refracted wave appears only in the region bounded by the earth and by a line through the origin having the direction of the second refracted wave.

Let us now combine all these results and apply to the incident configuration. We then have the following diagram for the total field configuration. We denote incident waves by I, reflected waves by RL, refracted waves by RR, and second refracted waves by S. Subscripts show which waves are associated. The barrier lines are denoted by B. (See Figure 4).

The descriptive behavior indicated in Figure 4 holds for all angles φ between zero and $\pi/2$, although the direction of the various waves and of the barrier lines depends on the angle φ. However, the behavior is quite different when the wave is incident from the right. In that case, the number of waves produced depends on the angle of the wedge. We shall discuss pictorially the behavior for a single wave incident from the right and traveling horizontally. Though such a wave cannot exist with our choice of



I_1 and I_2 - Incident Waves

RL_1 and RL_2 - Reflected Waves

B_1 and B_2 - Barriers of Reflected Waves

RR_1 and RR_2 - Refracted Waves

S_1 and S_2 - Second Refracted Waves

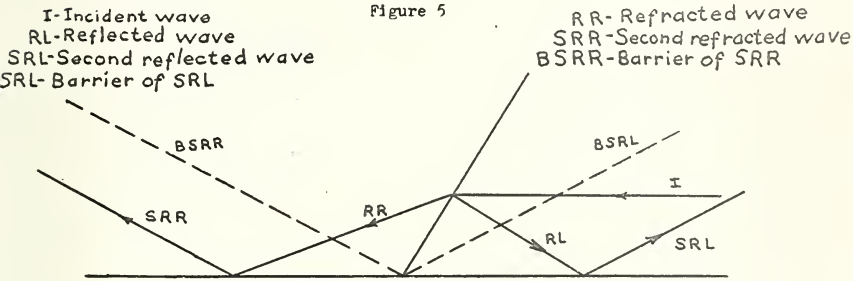
B_1' and B_2' - Barriers of Second Refracted Waves

Fig. 4

polarization, it can exist when H has only a z-component, the alternative polarization. Because the pictures would become much too cluttered if we drew the two incident waves and their behavior, as in Figure 4, we shall choose this simpler case of just one wave.

We shall now draw pictures of the reflections and refractions for several ranges of wedge angle. First suppose the angle α is between $\pi/2$ and $\pi/3$. Then we have the behavior indicated in Figure 5:

Figure 5

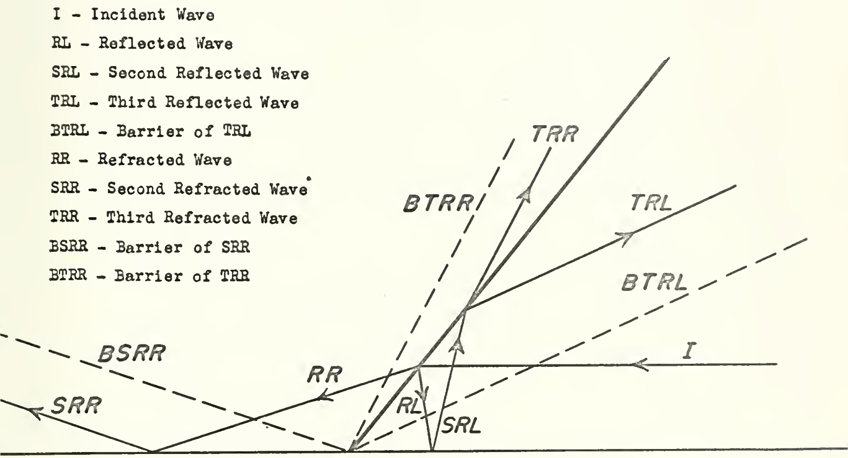


The incident wave is reflected backward and downward at the interface, so it strikes the ground and a second reflected wave is generated. This second wave will not strike the interface as long as α is greater than $\pi/3$. At $\alpha = \pi/3$, the second wave is easily seen to be parallel to the interface. The refracted wave, being bent away from the normal, is bent downward to strike the ground and produce a second refracted wave. The second reflected and second refracted waves both only exist between the earth and their respective barrier lines.

Next, suppose the wedge angle is between $\pi/3$ and $\pi/4$. Then the second reflected wave will strike the interface a second time. There, it

generates two more waves, both traveling to the right, one a reflected wave, the other refracted, neither of which strikes the ground again. This is shown in Figure 6.

Figure 6



I - Incident Wave
RL - Reflected Wave
SRL - Second Reflected Wave
TRL - Third Reflected Wave
BTRL - Barrier of TRL
RR - Refracted Wave
SRR - Second Refracted Wave
TRR - Third Refracted Wave
BSRR - Barrier of SRR
BTRR - Barrier of TRR

As can be easily seen, each time the wedge angle α passes through π/n, $n = 2,3,4, \ldots$, new waves are generated. If n is odd, the new waves are produced by a reflected wave striking the interface and undergoing partial reflection and refraction. If n is even, a reflected wave will strike the ground and produce an additional reflected wave. Thus, if α lies between π/n and $\pi/(n+1)$, there will be n waves present in the right hand region, not counting the incident wave, while in the left hand region there will be $\left[1/2(n+3)\right]$ which is the greatest integer contained in $1/2(n+3)$. We thus see that when the wedge angle is small, the wave pattern is very complicated.

The amplitudes of the waves can also be determined. Suppose the change in dielectric constant is small, so we write $z = (\epsilon_2 - \epsilon_1)/\epsilon_1$, where subscript one denotes the left hand side, subscript two the right hand side, and z is positive by assumption. Then we may classify a wave by the leading power of z in the power series expansion of its amplitude. For z small, a wave produced by a reflection at the interface has an amplitude one order higher than the wave the produces it. Assume the incident wave is of order unity.

In Figure 4, waves RL_1 and RL_2 are of order z, while waves RR_1, RR_2, S_1, S_2 are of order unity. In Figure 5, waves RL and SRL are of order z, waves RR and SRR of order unity. In Figure 6, waves RL and SRL, and TRR are of order z, TRL of order z^2, and waves RR and SRR of order unity. It is clear how the process continues.

In the next section we shall consider how the simple geometrical picture presented in this section is modified by diffraction effects. We shall discuss how the waves interfere and what is the actual total field amplitude at any point in a later section.

3. Solution of the Field Equations

In order to find the total fields, including the effects of diffraction, instead of just the plane-wave parts, we begin with Maxwell's equations:

$$\begin{aligned} &\nabla \times \vec{E} - i\omega\mu \vec{H} = 0 \\ &\nabla \times \vec{H} + i\omega\epsilon \vec{E} = 0 \end{aligned} \tag{3.1}$$

Here the time dependence $\exp(-i\omega t)$ is implied. We are interested in the case where $\vec{E}$ has only a z-component, so $\vec{E} = (0,0,E)$. The first set of Maxwell's equations then gives:

$$\begin{aligned} H_x &= \frac{1}{i\omega\mu}\frac{\partial E}{\partial y} \\ H_y &= -\frac{1}{i\omega\mu}\frac{\partial E}{\partial x} \\ H_z &= 0 \end{aligned} \tag{3.2}$$

while the second set yields the wave equation:

$$(3.3) \qquad (\nabla^2 + k^2)\, E = 0 \qquad k^2 = \omega^2 \mu \varepsilon$$

We are concerned with the geometry of Figure 1, and E has no z dependence. However, k is different in the two regions. Calling the left hand region 1 and the right hand region 2, we have $k = k_1$ in region 1, $k = k_2$ in region 2, where by the physical conditions $k_1 < k_2$.

In addition to the wave equation 3.3, we need a set of boundary conditions. These are determined by requiring that the tangential electric field shall vanish at the ground, which is taken to be a perfect conductor, and that the tangential components of electric and magnetic field shall be continuous upon crossing the interface. Let us denote by $\alpha+$ an angle very slightly greater than α, and by $\alpha-$ an angle very slightly less than α. The boundary conditions are then given by:

$$(3.4) \qquad E = 0 \qquad y = 0$$

$$E(\alpha+) = E(\alpha-)$$

$$\frac{\partial E}{\partial \theta}(\alpha+) = \frac{\partial E}{\partial \theta}(\alpha-)$$

The solution is then determined uniquely by the requirement that it have as incoming part (2.1), where it will be convenient to take $E_0 = 1$, and the remaining part of E shall represent outgoing waves. We shall now develop a method of solution of these equations.

The wave equation (3.3) may be rewritten in the forms:

$$(3.5) \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2\right) E = -\,(k_2^2 - k_1^2)\, S(\theta) E$$

$$S(\theta) = 1 \qquad 0 < \theta < \alpha$$

$$= 0 \qquad \alpha < \theta < \pi$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_2^2\right)E = \left(k_2^2 - k_1^2\right)\left[1 - S(\theta)\right] E \tag{3.6}$$

The form (3.5) will be convenient for treating reflected waves, the form (3.6) for transmitted waves. In each form, the expression on the right hand side is proportional to $k_2^2 - k_1^2 = p$. Hence, if this parameter p is small, we may treat the right hand sides as perturbation terms. We may therefore write:

$$E(x,y) = \sum_{0}^{\infty} p^n E_n(x,y) \tag{3.7}$$

It is clear that the term n = 0 represents the total field when the two regions have the same dielectric constant, and thus it equals the incident field. For the higher order terms, we substitute the expansion (3.7) into the wave equations (3.5) and (3.6) and equate like powers of p on both sides. There results:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_1^2\right)E_n(x,y) = - S(\theta)E_{n-1}(x,y) \tag{3.8}$$

$$n = 1,2,\cdots$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_2^2\right)E_n(x,y) = \left[1 - S(\theta)\right] E_{n-1}(x,y) \tag{3.9}$$

Thus, the functions E_n satisfy wave equations with forcing terms. The boundary conditions (3.4) also must be satisfied separately by each of the functions E_n. The additional requirement that E_n represents outgoing waves at infinity for $n > 0$ then determines E_n uniquely.

To solve the equations (3.8) and (3.9), we introduce the Green's functions:

$$G_1\,(x,x',y,y') = \frac{i}{4}\left[H_0^{(1)}\left(k_1\sqrt{(x-x')^2+(y-y')^2}\right) - H_0^{(1)}\left(k_1\sqrt{(x-x')^2+(y+y')^2}\right)\right]$$

(3.10)

$$G_2\,(x,x',y,y') = \frac{i}{4}\left[H_0^{(1)}\left(k_2\sqrt{(x-x')^2+(y-y')^2}\right) - H_0^{(1)}\left(k_2\sqrt{(x-x')^2+(y+y')^2}\right)\right]$$

These functions satisfy the differential equation:

(3.11)
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_{1,2}^2\right)G_{1,2} = -\delta(x-x')\;\delta(y-y')$$

using an obvious shorthand. Also, both G_1 and G_2 vanish at $y = 0$ and behave like outgoing waves at infinity.

We now shall use Green's theorem to solve (3.8). The solution of (3.9) proceeds in an equivalent manner. Green's theorem may be stated in the form:

$$\int_0^\infty\int_{-\infty}^\infty dx'dy'\left[E_n(x',y')(\nabla'^2+k_1^2)G_1(x,x',y,y')-G_1(x,x',y,y')(\nabla'^2+k_1^2)E_n(x',y')\right]$$

(3.12)

$$= \int_c dl'\left[E_n(x',y')\frac{\partial}{\partial n'}G_1 - G_1\frac{\partial}{\partial n'}E_n\right]$$

Here c denotes a contour consisting of the x axis and a large semicircle at infinity. Because of the boundary conditions, the integral along the x axis vanishes, and the outgoing waves condition insures the vanishing of the integral over the semicircle. Using the differential equations for E_n and G_1 gives:

$$0 = \int_0^\infty\int_{-\infty}^\infty dx'dy'\left[-E_n(x',y')\delta(x-x')\delta(y-y')+G_1(x,x',y,y')S(\theta')E_{n-1}(x',y')\right]$$

(3.13)

$$E_n(x,y) = \iint_2 dx'dy'\;E_{n-1}\,(x',y')\;G_1(x,x',y,y')$$

Here we have denoted the range of integration by the subscript 2, indicating it extends over region 2. It follows from (3.13) that if we know E_{n-1}, we can find E_n by an integration. Since E_o is known, we can find the various E_n's in succession. However, the labor of finding E_n for $n > 1$ becomes prodigious. But it will develop that E_1 displays the same geometrical behavior as the complete solution, save for certain amplitude coefficients, which are correct to first order. Hence, we shall restrict ourselves to the consideration of the term E_1, which will provide all the information we need.

A word of caution must be stated. The function F_n as determined from (3.13) satisfies the source-free wave equation in region 1 only. Hence in region 1 it may be expressed in terms of waves of the correct propagation constant. However, in region 2 the propagation constants will be incorrect, and we will commit considerable errors if we investigate the far fields in region 2 on the basis of (3.13). It is for this purpose that we developed the alternative representation characterized by (3.9). The solution of (3.9) analogous to (3.13) is:

$$E_n(x,y) = - \iint_1 dx'dy' \quad E_{n-1}(x',y') \quad G_2(x,x'y,y') \tag{3.14}$$

This expression is suitable for analyzing the fields in region 2.

An alternative method of treatment consists in applying Green's theorem directly to the differential equation (3.5). This will convert the differential equation into an integral equation. Thus, by analysis virtually identical with that used to derive (3.13) from (3.8), we obtain the integral equation:

$$E(x,y) = p \iint_2 dx'dy' \quad E(x',y') \quad G_1(x,x',y,y') + E_o(x,y) \tag{3.15}$$

The second term represents the incident wave, that is, that due to the sources at infinity. The first term is the analogue of a mass distribution in potential theory, or of a distribution of simple sources in acoustics. Such a distribution is known to give rise to a field which is continuous upon crossing the boundary of the region, and whose normal derivative is also continuous there. This field consequently satisfies the conditions of the problem.

We may view equation (3.15) as constituting an integral equation for the field E when the point x,y is in region 2. Once this equation is solved, we may use (3.15) to evaluate the field at points in region 1, treating (3.15) as the definition of the field for such points. To solve the integral equation, the method of successive iterations may be employed, regarding the integral term on the right as a correction term. This is equivalent to the Neumann series expansion of the solution. Thus we have as a first iteration $E = E_0$, as a second iteration:

$$(3.16) \qquad E(x,y) = E_0(x,y) + p \iint_2 dx' dy' E_0(x',y') G_1(x,x',y,y')$$

and so forth. This iteration scheme, which at each step substitutes the value of E from the previous step, clearly leads to the same answer as the perturbation method described earlier.

4. The Reflected Field

We shall now evaluate the integral (3.13) to find the reflected field to the first approximation. The integral we need is:

$$(4.1) \quad E_1 = \iint_2 dx' dy' G_1(x,x',y,y') e^{ik_1 x' \cos\varphi} \left[e^{ik_1 y' \sin\varphi} - e^{-ik_1 y' \sin\varphi} \right]$$

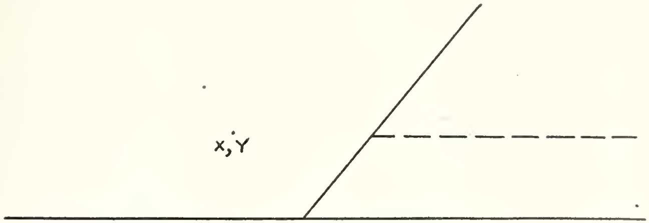
The evaluation of this integral is based upon the integral representation:(4)

$$H_0^{(1)}(k_1\sqrt{x^2+y^2}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{du}{\sqrt{k_1^2-u^2}}\, e^{iu|x|+i\sqrt{k_1^2-u^2}|y|} \tag{4.2}$$

in which it is assumed that k_1 has a positive imaginary part.* By means of this integral, we can work out the x', y' integrations of (4.1), leaving the integration over u to the end. We will then obtain far field approximations by application of the principle of stationary phase. We shall write $\psi = \sqrt{k_1^2-u^2}$ and drop the subscript on k_1.

The integrand (4.2) changes form whenever x or y passes thru zero. Hence, we will have to break up (4.1) into several regions. Let us suppose the observation point x,y is in the left hand range with x negative. The break-up is indicated in Figure 7.

Figure 7



We are dealing with the simplest possible case, since there are only two ranges of integration, those with $y' < y$ and with $y' > y$.

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(4) G. N. Watson, "Theory of Bessel Functions", Macmillan 1948, p. 416

* The absolute value sign on x is unnecessary, but we use the form 4.2 to avoid modification of the position of certain subsequently occurring singularities in the complex u-plane.

Thus we have upon substituting (4.2) into (4.1):

$$(4.3)\quad E_1 = \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{du}{v}\left[\begin{array}{l}\displaystyle\int_0^{y} dy'\int_{y'\cot\alpha}^{\infty} dx'\, e^{iu(x'-x)}\left[e^{iv(y-y')} - e^{iv(y+y')}\right] e^{ikx'\cos\varphi}\left[e^{iky'\sin\varphi} - e^{-iky'\sin\varphi}\right] \\ \displaystyle -\int_y^{\infty} dy'\int_{y'\cot\alpha}^{\infty} dx'\, e^{iu(x'-y)}\left[e^{iv(y'-y)} - e^{iv(y+y')}\right] e^{ikx'\cos\varphi}\left[e^{iky'\sin\varphi} - e^{-iky'\sin\varphi}\right]\end{array}\right]$$

The integration over x' and y' now involves only exponential functions. The algebra is rather tedious, so rather than go thru the details we shall only give the result. Hence, after the integrations and simplifications, we obtain:

$$(4.4)\quad E_1 = \frac{\sin^2\alpha}{2\pi i}\int_{-\infty}^{\infty}\frac{du\, e^{-iux}}{(u+k\cos\varphi)^2}\left[\begin{array}{l}\displaystyle e^{ivy}\left\{\frac{1}{u+k\cos(2\alpha-\varphi)} - \frac{1}{u+k\cos(2\alpha+\varphi)}\right\} \\ \displaystyle - e^{i(u+k\cos\varphi)y\cot\alpha}\left\{\frac{e^{iky\sin\varphi}}{u+k\cos(2\alpha-\varphi)} - \frac{e^{-iky\sin\varphi}}{u+k\cos(2\alpha+\varphi)}\right\}\end{array}\right]$$

This integal represents a superposition of plane waves of all directions of propagation. Explicit evaluation requires examination of the singularities of the integrand in the complex u-plane, which is the next order of business.

The integrand of (4.4) is written as the sum of two terms, one involving exp ivy and the other independent of v. The second of these may be evaluated directly. This term has a double pole at $u = -k\cos\varphi$ and simple poles at $u = -k\cos(2\alpha+\varphi)$, $u = -k\cos(2\alpha-\varphi)$, and no other singularities in the finite part of the complex u-plane. Also, because $-x+y\cot\alpha$ is positive for x negative, the path of integration may be closed by an infinite semicircle in the upper half-plane and the integral over the semicircle tends to zero as the radius increases. Hence, the integral is equal to the sum of the residues at the poles in the upper half-plane. We now recall that the imaginary part of k is positive. For the incident wave to travel to the right, $0 \leq \varphi \leq \frac{\pi}{2}$, and hence $-k\cos\varphi$ has a negative imaginary

part and the double pole is outside the contour of integration. Let us now suppose that the angle of incidence is such that $2\alpha - \varphi$ and $2\alpha + \varphi$ are angles in the second quadrant. If α is fairly close to $\frac{\pi}{2}$ these conditions are satisfied for a wide range of ϕ. The cosines of $2\alpha - \varphi$ and $2\alpha + \varphi$ will then be negative, and the integrand will have two poles in the upper half-plane. Evaluating the residues at these poles now gives:

$$E_{12} = -\frac{\sin^2\alpha}{k^2}\left[\frac{1}{[\cos\varphi - \cos(2\alpha-\varphi)]^2}\, e^{ik[\{\cos\varphi - \cos(2\alpha-\varphi)\}\cot\alpha + \sin\varphi]y + ik\cos(2\alpha-\varphi)x} - \frac{1}{[\cos\varphi - \cos(2\alpha+\varphi)]^2}\, e^{ik[\{\cos\varphi - \cos(2\alpha+\varphi)\}\cot\alpha - \sin\varphi]y + ik\cos(2\alpha+\varphi)x}\right] \tag{4.5}$$

Simplifying the amplitudes and exponents yields:

$$E_{12} = -\frac{1}{4k^2}\left[\frac{1}{\sin^2(\alpha-\varphi)}\, e^{ik[x\cos(2\alpha-\varphi) + y\sin(2\alpha-\varphi)]} - \frac{1}{\sin^2(\alpha+\varphi)}\, e^{ik[x\cos(2\alpha+\varphi) + y\sin(2\alpha+\varphi)]}\right] \tag{4.6}$$

Upon consulting Figure 4, we see that this term represents two plane waves, in the directions of RL_1 and RL_2 respectively. The amplitudes of these waves thus have been obtained correct to first order in $k_2^2 - k_1^2$. We thus have the reflection coefficients correct to first order in $k_2^2 - k_1^2$. These may be directly shown to be exactly the Fresnel reflection coefficients, again to the same order. Thus, the reflected plane waves are the same as those produced by an infinite dielectric medium with the appropriate angles of incidence. This is reasonable, since the plane wave part of the solution is mainly determined by the degree of approximation of the interface to a plane extending upward to infinity, rather than by the behavior near the ground. We may expect that the actual reflection coefficient will

be the same as the Fresnel coefficient, correct to all orders. However, E_{12} only represents a part of the total reflected field. The remainder is given by the term involving e^{ivy} in (4.4), which we shall call E_{11} and will proceed to investigate. E_{11} has poles at the same values of u as the poles of E_{12}. However, it also has branch points at u = k and u = - k, because of the appearance of $v = \sqrt{k^2-u^2}$. Hence, we cannot close the contour and evaluate residues, since there will remain a contribution from the branch cut that must be put into the u-plane to render the integrand single-valued. To evade this difficulty, we shall evaluate the integral approximately by use of the principle of stationary phase. The resulting expression will represent E_{11} provided the distance from the origin is sufficiently large, and will be fairly accurate even down to within a wave-length of the origin.

A point of stationary phase exists where the derivative of the exponent vanishes. Writing x = r cos Θ, y = r sin Θ, where Θ is an angle in the second quadrant, the phase becomes:

(4.7)
$$f(u) = r\left[-u \cos \Theta + \sqrt{k^2-u^2} \sin \Theta\right]$$

Differentiating with respect to u and setting the derivative equal to zero gives the equation:

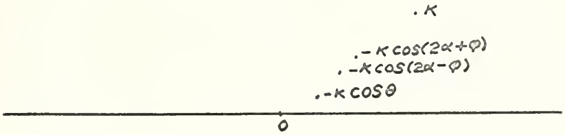
(4.8)
$$0 = - \cos \Theta - \frac{u_o}{\sqrt{k^2-u_o^2}} \sin \Theta = f'(u_o)$$

A solution of this is $u_o = - k \cos \Theta$, and it is easily seen that this is the only solution. Because Θ is in the second quadrant, u_o is in the upper half-plane. We also have:

(4.9)
$$f(u_o) = kr$$
$$f''(u_o) = - \frac{r}{k \sin^2\Theta}$$

We now must consider the disposition of the poles and the stationary point in the complex u-plane. First assume $2\alpha + \varphi > 2\alpha - \varphi > \theta$. Then $-\cos(2\alpha+\varphi) > -\cos(2\alpha-\varphi) > -\cos\theta$, and we have Figure 8:

Figure 8



In this case, we may displace the contour of integration from the real axis to the line Im u = Im (-k cos θ) without crossing any poles of the integrand. Calling the factor multiplying the exponential g (u), we have:

$$E_{11} = \int_{\overline{u_o}} g(u)\, e^{if(u)}\, du \tag{4.10}$$

Now f(u) has a strong maximum at $u = u_o$. Hence, most of the contribution to the integral comes from the neighborhood of $u = u_o$, since the exponential oscillates rapidly at other points and interference makes the total contribution away from $u = u_o$ small. We may accordingly write:

$$E_{11} \sim g(u_o)\, e^{if(u_o)} \int_{-\infty}^{\infty} e^{\frac{1}{2} i f''(u_o)(u-u_o)^2}\, du \tag{4.11}$$

where the higher order terms have been dropped. This integral is a standard form, and we thus have:

$$E_{11} \sim g(u_o)\, e^{if(u_o)} \frac{\sqrt{2\pi}}{\sqrt{|f''(u_o)|}}\, e^{-i\frac{\pi}{4}} \tag{4.12}$$

Hence, in the range $2\alpha+\varphi>2\alpha-\varphi>\theta$, we have approximately:

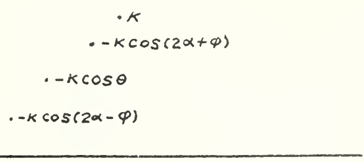
$$E_{11} \sim \frac{e^{ikr}}{\sqrt{kr}}\,\frac{e^{i\frac{1}{4}\pi}}{\sqrt{2\pi}}\,\frac{\sin^2\alpha}{k^2}\,\frac{\sin\theta}{(\cos\varphi-\cos\theta)^2}\left[-\frac{1}{\cos(2\alpha-\varphi)-\cos\theta}+\frac{1}{\cos(2\alpha+\varphi)-\cos\theta}\right] \tag{4.13}$$

This represents an outgoing cylindrical wave, with a rather complicated angular pattern. Hence, for r sufficiently large, we may omit this cylindrical wave, and the whole field is given by (4.6). The range of validity of this result is the region between the barrier line B_1 (see Figure 4), and the y-axis, because of the assumption $x<0$. However, it may be shown by direct manipulation that (4.4) represents E_1 throughout the range $\theta>\alpha$, which extends the region of validity up to the wedge interface. The proof requires evaluating the integral (4.1) with x positive, but above the wedge, and differs only in some algebraic details from that used to obtain (4.4). Therefore, we see that the exact theory agrees with the geometrical optics result.

The formula (4.13) represents E_{11} only if we are not too close to the barrier line $\theta = 2\alpha-\varphi$, since there the amplitude of the cylindrical wave becomes infinite. This implies that the field changes character upon crossing the barrier. We shall investigate this point more closely in a later section.

Next, suppose that $2\alpha+\varphi>\theta>2\alpha-\varphi$, so the observation point is between the barrier lines. Then the singularities are arranged as in Figure 9..

Figure 9


κ
$-\kappa\cos(2\alpha+\varphi)$
$-\kappa\cos\theta$
$-\kappa\cos(2\alpha-\varphi)$

Now, in order to displace the contour to pass thru the stationary point, we must cross the pole at $u = -k \cos(2\alpha - \varphi)$. Hence, the value of E_{11} is found by adding the residue at the pole to (4.13). This residue is found directly to be:

$$(4.14) \qquad \frac{\sin^2\alpha}{k^2\left[\cos\varphi - \cos(2\alpha-\varphi)\right]^2} e^{+ik\cos(2\alpha-\varphi)x + ik\sin(2\alpha-\varphi)y}$$

$$= \frac{1}{4k^2} \frac{1}{\sin^2(\alpha-\varphi)} e^{ik\left[x\cos(2\alpha-\varphi)+y\sin(2\alpha-\varphi)\right]}$$

This residue will just cancel the plane wave RL_1 in E_{12}. Hence, in this region we have for the total field the plane wave RL_2 plus the cylindrical wave, again agreeing with geometrical optics. This holds provided we are not too near B_1 and B_2.

Finally, suppose $\theta > 2\alpha + \varphi$. Then we see that we must add the residues at both poles to (4.13). The residue at the pole $u = -k\cos(2\alpha+\varphi)$ is:

$$(4.15) \qquad -\frac{\sin^2\alpha}{k^2\left[\cos\varphi - \cos(2\alpha+\varphi)\right]^2} e^{ik\left[x\cos(2\alpha+\varphi) + y\sin(2\alpha+\varphi)\right]}$$

$$= -\frac{1}{4k^2} \frac{1}{\sin^2(\alpha+\varphi)} e^{ik\left[x\cos(2\alpha+\varphi) + y\sin(2\alpha+\varphi)\right]}$$

This cancels the plane wave RL_2 in E_{12}, so in this range the entire field is given by the cylindrical wave.

Consequently, we see that the first order field E_1 displays the behavior of the geometrical optics solution, since we have obtained a wave picture for E_1 which is the same as that given by geometrical optics. Moreover, the cylindrical wave portion of the total field, given by (4.13), shows that the geometrical optics solution is only valid at considerable distances from the origin, because of the requirements under which the stationary phase analysis represents a good approximation to the integral E_{11}. Nevertheless, the stationary phase method is only off by about 10% when we are one wavelength from the origin, so the combination of plane and cylindrical waves is an accurate representation of the field over most of region 1,

except in the immediate neighborhood of the barrier lines, where the plane wave is discontinuous and the cylindrical wave is infinite. We shall now discuss the transmitted field, and then consider the field near barriers.

5. The Transmitted Field

The transmitted field is much more complicated to calculate than the reflected field. Since the imaginary part of k is taken to be positive, the field E_o is infinite at x equals minus infinity, and consequently the integrals do not converge. In order to obtain results, we must choose a form for E_o which differs from (2.1) at infinity, but is close to the correct incident field near the interface.* The integrals will then be convergent. We shall assume that E_o, instead of being an incoming plane wave at infinity, has a change of character at $x = x_o$, where x_o is large and negative, in that it shall be an outgoing wave for $x < x_o$. We thus choose for E_o:

$$E_o = e^{ik_1\cos\varphi|x-x_o|}\left[e^{ik_1y\sin\varphi} - e^{-ik_1y\sin\varphi}\right] \tag{5.1}$$

At the end of the calculation we shall let x_o tend to minus infinity. We note that for $x > x_o$, E_o as chosen here differs from (2.1) only in a phase factor $e^{-ik_1\cos\varphi\, x_o}$, and hence has the same space dependence as (2.1). We now insert this function into (3.14) to compute E_1. Upon employing the integral representation (4.2) with $v = \sqrt{k_2^2 - u^2}$, we have for E_1:

$$E_1 = -\frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{du}{v}\iint_1 dx'dy'\, e^{ik_1\cos\varphi|x'-x_o|+iu|x-x'|}\left(e^{iv|y-y'|} - e^{iv(y+y')}\right)\left(e^{ik_1y'\sin\varphi} - e^{-ik_1y\sin\varphi}\right) \tag{5.2}$$

The point x,y is in region 2, so we have to break up the integral as in Figure 10.

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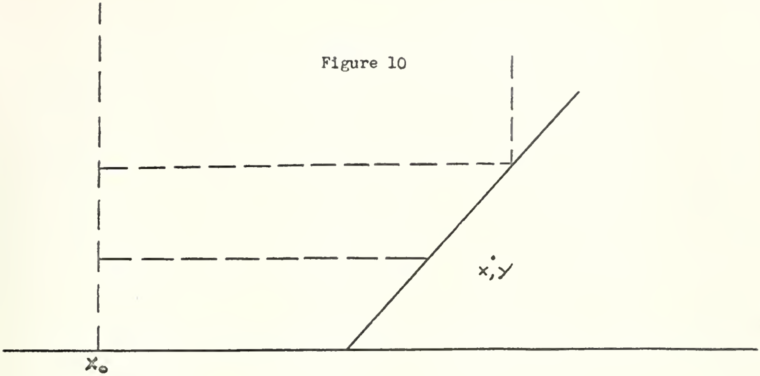
* This may be avoided by breaking the range of integration into that for x' negative and x' positive, and using to first order the known solution of the problem of reflection and refraction at a plane interface to interpret the divergent x' negative integration.

Figure 10



There accordingly results for the integral over x' and y' in (5.2), which we shall call I, upon omitting the lengthy intervening algebra,

$$
\begin{aligned}
I = &-\frac{2k_1\cos\varphi}{u^2-k_1^2\cos^2\varphi}\, e^{iu(x-x_o)}\left(e^{ik_1y\sin\varphi}-e^{-ik_1y\sin\varphi}\right)\frac{2v}{k_1^2\sin^2\varphi-k_2^2+u^2} \\
&+\frac{e^{-ik_1\cos\varphi x_o+iux+i(k_1\cos\varphi-u)\cot\alpha}{}_{2v\sin^2\alpha}}{u-k_1\cos\varphi}\left[\frac{e^{ik_1y\sin\varphi}}{(u-k_1\cos\varphi)(u-k_1\cos(2\alpha-\varphi))-(k_2^2-k_1^2)\sin^2\alpha} - \frac{e^{-ik_1y\sin\varphi}}{(u-k_1\cos\varphi)(u-k_1\cos(2\alpha+\varphi))-(k_2^2-k_1^2)\sin^2\alpha}\right] \\
&-\frac{e^{-ik_1\cos\varphi x_o+iux+ivy}{}_{2v\sin^2\alpha}}{u-k_1\cos\varphi}\left[\frac{1}{(u-k_1\cos\varphi)(u-k_1\cos(2\alpha-\varphi))-(k_2^2-k_1^2)\sin^2\alpha} - \frac{1}{(u-k_1\cos\varphi)(u-k_1\cos(2\alpha+\varphi))-(k_2^2-k_1^2)\sin^2\alpha}\right]
\end{aligned}
\tag{5.3}
$$

This expression when divided by v represents the integrand of (5.2), from which we are to calculate E_1. Now suppose we set $k_1 = k_2$ here; which would lead to no errors in first order. The integrand would then have a double pole at $u = k_1 \cos\varphi$. When complex integration methods are used to evaluate E_1 in terms of residues, the contribution from the double pole would be of order xe^{ikx}, and thus would represent a plane wave of increasing amplitude. Such a wave, of course, cannot be generated, whence it follows that it is a mathematical fiction produced by an inconsistent set of approximations. We thus keep the term $(k_2^2 - k_1^2)\sin^2\alpha$ in the denominators of (5.3), whence the pole at $u = k_1 \cos\varphi$ appears to be only of first order. Actually the several terms of (5.3) combine in such a way that the integrand is regular at $u = k_1 \cos\varphi$. Accordingly, we may omit the consideration of the pole $u = k_1 \cos\varphi$ in the separate terms of (5.3), since they will cancel upon combination.

The three terms of (5.3) have the following physical interpretation. The first term cancels the incident wave. The second term represents the refracted waves, and the third term provides the second refracted waves and also the cylindrical waves which represent the diffracted part of the field. Proof of these statements will be presented in the succeeding analysis.

The two poles of the first term are located at $u = \pm\left[k_2^2 - k_1^2 \sin^2\varphi\right]^{\frac{1}{2}}$ We shall assume that the imaginary parts of k_1 and k_2 are in the same ratio as the real parts, so k_1 and k_2 are complex numbers with the same phase. Then the only pole of E_{11} in the upper half plane is at $u_{11} = \left[-k_1^2 \sin^2\varphi + k_2^2\right]^{\frac{1}{2}}$, and we may close the contour and evaluate by residues, obtaining:

$$(5.4) \qquad E_{11} = -\frac{2k_1\cos\varphi}{(k_2^2 - k_1^2)}\, e^{iu_{11}(x-x_o)}\left(e^{ik_1 y \sin\varphi} - e^{-ik_1 y \sin\varphi}\right) \frac{1}{2\left[k_1^2\cos^2\varphi + k_2^2 - k_1^2\right]^{\frac{1}{2}}}$$

For small values of $p = k_2^2 - k_1^2$, this is approximately:

(5.5)
$$E_{11} \sim -\frac{1}{k_2^2-k_1^2}\, e^{ik_1\cos\varphi(x-x_0)}\left(e^{ik_1 y\sin\varphi} - e^{-ik_1 y\sin\varphi}\right)$$

When this is substituted into the expansion $E = \sum_0^\infty p^n E_n$, it just cancels the term n=0. This is proper, since the incident wave should be removed to leave only the refracted and diffracted fields, which are all that are present in region 2.

The integrands of E_{12} and E_{13} each have four poles. In addition, E_{13} has a branch point at $u = k_2$, because of the term involving v in the exponent. Hence, we must apply the stationary phase method to E_{13}, to obtain a good approximation. However, E_{12} may be evaluated explicitly in terms of residues.

(5.6) The poles of E_{12} are located at:

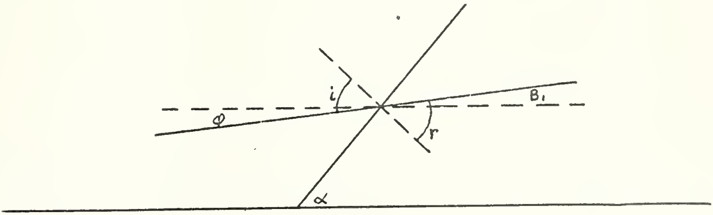
$$u_{21} = k_1\cos\alpha\cos(\alpha-\varphi) + \sin\alpha\left[k_1^2\sin^2(\alpha-\varphi) + k_2^2 - k_1^2\right]^{\frac{1}{2}}$$
$$u_{22} = k_1\cos\alpha\cos(\alpha+\varphi) + \sin\alpha\left[k_1^2\sin^2(\alpha+\varphi) + k_2^2 - k_1^2\right]^{\frac{1}{2}}$$
$$u_{23} = k_1\cos\alpha\cos(\alpha-\varphi) - \sin\alpha\left[k_1^2\sin^2(\alpha-\varphi) + k_2^2 - k_1^2\right]^{\frac{1}{2}}$$
$$u_{24} = k_1\cos\alpha\cos(\alpha+\varphi) - \sin\alpha\left[k_1^2\sin^2(\alpha+\varphi) + k_2^2 - k_1^2\right]^{\frac{1}{2}}$$

The square root is so defined that when $k_2 = k_1$, $\left[k_1^2\sin^2(\alpha-\varphi)\right]^{\frac{1}{2}} = k_1\sin(\alpha-\varphi)$, and similarly for $\left[k_1^2\sin^2(\alpha+\varphi)\right]^{\frac{1}{2}}$. Now when $k_2 = k_1$, we have the limiting values $u_{21} = u_{22} = k_1\cos\varphi$, $u_{23} = k_1\cos(2\alpha-\varphi)$, $u_{24} = k_1\cos(2\alpha+\varphi)$. Under the previous assumptions as to the values of the angles involved, u_{21} and u_{22} then have positive imaginary parts, while u_{23} and u_{24} have negative imaginary parts. This situation continues to hold for $k_2 > k_1$, since the imaginary parts of all the square roots increase. Hence, when we close the contour with a semicircle in the upper half plane, which contributes nothing because $x - y\cot\alpha$ is positive, (Region 2), only the poles u_{21} and u_{22} will lie within the contour. The integral E_{12} accordingly is the sum of the residues at the poles u_{21} and u_{22}, and is thus:

$$(5.7) \qquad E_{12} = \frac{1}{2} \sin\alpha e^{-ik_1 x_0 \cos\varphi} \left[\frac{e^{iu_{21}x + i\left[(k_1\cos\varphi - u_{21})\cot\alpha + k_1 \sin\varphi \right] y}}{(u_{21} - k_1\cos\varphi) \left[k_1^2 \sin^2(\alpha-\varphi) + k_2^2 - k_1^2 \right]^{\frac{1}{2}}} - \frac{e^{iu_{22}x + i\left[(k_1\cos\varphi - u_{22})\cot\alpha - k_1 \sin\varphi \right] y}}{(u_{22} - k_1\cos\varphi) \left[k_1^2 \sin^2(\alpha+\varphi) + k_2^2 - k_1^2 \right]^{\frac{1}{2}}} \right]$$

We shall now develop a geometric interpretation of the roots u_{21} and u_{22}. Since u_{22} differs from u_{21} only by changing the sign of φ, it will be sufficient to consider u_{21}. We shall show that $u_{21} = k_2 \cos B_1$, where B_1 is the angle describing the direction of propagation of the first refracted wave (RR_1, Fig. 4). Now if p is very small, the wave RR_1, which arises from an upgoing wave, (I_1), will still be upgoing after refraction. As p increases, it will become horizontal, and then downgoing. Thus, the sign of the angle B_1 will change as p increases. We shall define B_1 as positive for upgoing waves and negative for downgoing. Let us first consider the case B_1 positive. We then have Figure 11.

Figure 11



By using Snell's law and elementary geometry, we obtain the following sequence of relations:

(5.8)
$$\begin{aligned}
i &= \frac{\pi}{2} - (\alpha-\phi) \\
\sin i &= \cos(\alpha-\phi) \\
B_1 &= r - \left(\frac{\pi}{2} - \alpha\right) \\
\cos B_1 &= \sin(r+\alpha) \\
k_2 \sin r &= k_1 \sin i = k_1 \cos(\alpha-\varphi) \\
k_2 \cos B_1 &= k_2 \sin r \cos\alpha + k_2 \cos r \sin\alpha \\
&= k_1 \cos\alpha \cos(\alpha-\phi) + \sin\alpha\left[k_1^2 \sin^2(\alpha-\phi)+k_2^2-k_1^2\right]^{\frac{1}{2}} \\
&= u_{21}
\end{aligned}$$

Thus, the value of u_{21}, which from (5.7) is the horizontal component of the propagation vector of the wave, is the same as that of the wave RR_1, determined by geometrical optics. A virtually identical proof shows that when the wave is sufficiently refracted to be bent downwards, u_{21} still equals $k_2 \cos B_1$. We now must discuss the coefficient of y in the exponent. Since the field E_{12} satisfies the wave equation, this coefficient must have the magnitude $k_2|\sin B_1|$. However, the sign of this quantity must be found. We have by direct manipulation:

(5.9)
$$(k_1\cos\varphi-u_{21})\cot\alpha+k_1\sin\varphi=$$
$$k_1\sin\alpha\cos(\alpha-\phi) - \cos\alpha\left[k_1^2\sin^2(\alpha-\varphi) + k_2^2-k_1^2\right]^{\frac{1}{2}}$$

When $k_2 = k_1$, this expression equals $k_1 \sin\varphi$. As k_2 increases, the whole expression decreases, because of the increase of the square root, and it reaches zero at a value of p equal to $k_1^2\sin(2\alpha-\phi)\sin\varphi/\cos^2\alpha$. For larger values of p, (5.9) becomes negative. Hence, the coefficient equals $k_2 \sin B_1$, in accordance with the convention that B_1 is negative for down-going waves. Thus, the field E_{12} represents the refracted waves. We now expand the amplitudes of the waves in E_{12} to first order in p, and obtain the simple result:

(5.10)

$$E_{12} = \frac{e^{-ik_1x_0\cos\varphi}}{k_2^2 - k_1^2} \left[\begin{array}{c} e^{ik_2(x\cos B_1 + y\sin B_1)} \\ \\ - e^{ik_2(x\cos B_2 + y\sin B_2)} \end{array} \right]$$

Here the angle B_2 is that describing the direction of propogation of the wave RR_2, and is negative for all values of p.

Finally, we consider the field E_{13}. The integrand has poles in the upper half plane at u_{21} and u_{22}, a branch point at k_2, which prevents closing the contour, and a stationary phase point at $k_2 \cos\theta$. The stationary phase method may be applied to evaluate the integral, provided that the contour may be displaced to pass thru the stationary phase point. This requires investigation of the disposition of the critical points in the complex u-plane. Now $\mathrm{Im}\, u_{22} < \mathrm{Im}\, u_{21}$, since B_2 is clearly a larger (in magnitude) angle than B_1. Hence, we have three possibilities. First, $\mathrm{Im}\, k_2 \cos\theta < \mathrm{Im}\, u_{22}$, or $\theta > |B_2|$. Then the contour may be displaced to pass thru $k_2 \cos\theta$, and evaluation of the integral yields the cylindrical wave:

(5.11)

$$E_c = - \frac{\sin^2\alpha}{\sqrt{2\pi}} \frac{k_2 \sin\theta \, e^{-ik_1x_0\cos\varphi}}{k_2\cos\theta - k_1\cos\varphi} \frac{e^{i(k_2r - \frac{\pi}{4})}}{\sqrt{k_2r}} \left[\begin{array}{c} \dfrac{1}{(k_2\cos\theta - u_{21})(k_2\cos\theta - u_{23})} \\ \\ - \dfrac{1}{(k_2\cos\theta - u_{22})(k_2\cos\theta - u_{24})} \end{array} \right]$$

Hence, in the region $\alpha > \theta > |B_2|$, the field is composed of the refracted waves given by (5.10) and the cylindrical diffracted wave given by (5.11). Next, suppose $\mathrm{Im}\, u_{22} < \mathrm{Im}\, k_2\cos\theta < \mathrm{Im}\, u_{21}$, or $|B_2| > \theta > |B_1|$. Then we must add the residue at the pole u_{22} to E_c. This residue yields a plane wave whose horizontal component of propagation vector is $u_{22} = k_2\cos B_2$, but whose vertical component must

be positive, and is thus $k_2 |\sin B_2|$. Upon expanding the amplitude of this wave to first order in p one obtains:

(5.12)
$$\frac{e^{-ik_1 x_0 \cos\varphi}}{k_2^2 - k_1^2} \; e^{ik_2(x \cos B_2 + y |\sin B_2|)}$$

This wave is thus the same as the second refracted wave S_2. It appears only below the barrier line B_2, where $\theta = |B_2|$, exactly as given by geometrical optics. Furthermore, the sign of the amplitude of this wave is opposite to that of the corresponding wave in (5.10), thereby displaying the change in phase of 180° upon reflection from the ground.

Finally, suppose $\mathrm{Im}\, u_{21} < \mathrm{Im}\, k_2 \cos\theta$, or $\theta < |B_1|$. Then, in addition to the residue at u_{22}, we must add the residue at u_{21} to the cylindrical wave E_c. This residue is analogous to (5.12),

(5.13)
$$-\frac{e^{-ik_1 x_0 \cos\varphi}}{k_2^2 - k_1^2} \; e^{ik_2(x \cos B_1 + y |\sin B_1|)}$$

This wave appears only below the barrier line $\theta = |B_1|$, and corresponds to the second refracted wave S_1.

Now suppose that p is so small that B_1 is positive, so the wave RR_1 does not strike the ground. Then we find that for $\theta > \varphi$, the field is given by the refracted waves RR_1 and RR_2. For $\theta < \varphi$, the wave S_1 cancels the wave RR_1, but the wave S_2 is to this approximation just equal to RR_1, so there is no contradiction. For p sufficiently large that B_1 is negative, we have derived exactly the field given by geometrical optics and illustrated in Figure 4.

Upon inserting the several expressions into the expansion $E = \sum_0^\infty p^n E_n$, and dropping the phase factor $e^{-ik_1 x_0} \cos\varphi$ we have the following results:

(5.14) For $\alpha > \theta > |B_2|$

$$E = e^{ik_2(x \cos B_1 + y \sin B_1)} - e^{ik_2(x \cos B_2 + y \sin B_2)} + pE_c$$

For $|B_2| > \theta > |B_1|$

$$E = e^{ik_2(x \cos B_1 + y \sin B_1)} - e^{ik_2(x \cos B_2 + y \sin B_2)} + e^{ik_2(x \cos B_2 + y |\sin B_2|)} + pE_c$$

For $|B_1| > \theta$

$$E = e^{ik_2(x \cos B_1 + y \sin B_1)} - e^{ik_2(x \cos B_1 + y |\sin B_1|)}$$
$$- e^{ik_2(x \cos B_2 + y \sin B_2)} + e^{ik_2(x \cos B_2 + y |\sin B_2|)} + pE_c$$

According to (5.14), the transmission coefficients of the various waves are unity to the first order of approximation. Since the reflection coefficients are proportional to p, we would expect transmission coefficients behaving like $1-ap^2$, in agreement with conservation of energy. Thus, they should be unity to first order in p. Actually the transmission coefficients should be the same as the Fresnel coefficients, to all orders of approximation.

To sum up our considerations on the transmitted field, we have shown that the field given by the wave analysis agrees with that given by geometrical optics in every detail, and in addition we have found the diffracted field E_c. All these results, of course, are only valid when we are not too close to the barrier lines. We shall now develop the theory of the behavior of the fields near the barrier lines.

6. The Field Near Barrier Lines

In the neighborhood of a barrier line, the considerations which lead to the cylindrical waves have to be modified. We recall that we are dealing with an integral of the form:

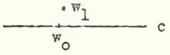
(6.1) $$\int_{-\infty}^{\infty} g(w)\ e^{if(w)} dw$$

where $f'(w)$ has a zero at $w = w_0$ and $g(w)$ is supposed to be slowly varying. The reason for the change from u to w will become clear later. The major contribution to the integral then comes from the neighborhood of $w = w_0$, and we replace $g(w)$ by $g(w_0)$ and extract $g(w_0)$ from under the integral sign. The remaining integral is then evaluated.

At the barrier lines, the stationary phase point of the integrand coincides with a pole. We then have $g(w_0) = \infty$, and the formula (4.12) certainly cannot be expected to be valid. A modification of the procedure, developed by Pauli(5), and Ott(6), enables sensible results to be obtained. We shall present this modification here.

Suppose the integrand of (6.1) has a pole at $w = w_1$, where w_1 is close to w_0. What we mean by "close" will be made more precise later. We than may write $g(w) = h(w)/(w-w_1)$, where we assume $h(w_1)$ to be slowly varying. This last assumption is equivalent to stating that the poles of $g(w)$ are not close together, so that the maxima of the integrand do not interfere with each other. We then will replace $h(w\ )$ by $h(w_0)$, extract it from the integral, and forget about it, since it plays no part in the further calculations. We shall also assume that w_0 and w_1 are in the upper half-plane, with $\mathrm{Im}\ w_1 > \mathrm{Im}\ w_0$. The case $\mathrm{Im}\ w_1 < \mathrm{Im}\ w_0$ will be discussed later. We may then displace the path to pass thru w_0, so we must deal with an integral of the form:

(6.2) $$J = \int_c e^{if(w)} \frac{dw}{w-w_1}$$



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(5) W. Pauli, Phys. Rev. 54, 924 (1938)
(6) H. Ott, Ann. die Phys. 43, 393 (1943)

The major contribution still comes from the neighborhood of w_o, and we may expand the exponent, which yields upon keeping quadratic terms:

(6.3)
$$J = e^{if(w_o)}\int_c e^{\frac{1}{2}if''(w_o)(w-w_o)^2} \frac{dw}{w-w_1}$$
$$= e^{if(w_o)}\int_{-\infty}^{\infty} e^{\frac{1}{2}if''(w_o)w^2} \frac{dw}{w-(w_1-w_o)}$$

Under the assumption that f(w) has a strong maximum at $w = w_o$, $f''(w_o)$ is negative and large compared to unity. The integral may be transformed into:

(6.4)
$$J = e^{if(w_o)}\int_{-\infty}^{\infty} e^{-iw^2} \frac{dw}{w-\sqrt{\frac{|f''(w_o)|}{2}}\,(w_1-w_o)}$$
$$= e^{if(w_o)}\int_{-\infty}^{\infty} e^{-iw^2} \frac{dw}{w-z}$$

Here we have introduced a dimensionless parameter z, defined by $z = \sqrt{\frac{|f''(w_o)|}{2}}\,(w_1-w_o)$. Although $f''(w_o)$ is large, z may take on any value if w_1-w_o varies. Hence, we shall investigate (6.4) for z arbitrary. We note that the imaginary part of z is positive. Small values of z correspond to "closeness" of the pole to the stationary point, clarifying the meaning of w_1 close to w_o.

We split the integral (6.4) into the two parts from 0 to ∞ and from -∞ to 0, and write w = - w in the second. Then:

(6.5)
$$J = e^{if(w_o)}\left[\int_0^{\infty} e^{-iw^2} \frac{dw}{w-z} - \int_0^{\infty} e^{-iw^2} \frac{dw}{w+z}\right]$$
$$= e^{if(w_o)}\int_0^{\infty} e^{-iw^2} \frac{2z}{w^2-z^2}\,dw$$
$$= e^{if(w_o)}\, 2ze^{-iz^2}\int_0^{\infty} e^{-i(w^2-z^2)} \frac{dw}{w^2-z^2}$$

We introduce a real positive parameter t into the exponent, and write:

$$F(t,z) = \int_0^\infty e^{-it(w^2-z^2)} \frac{dw}{w^2-z^2} \tag{6.6}$$

The integral in (6.5) equals F(1,z). Differentiating F with respect to t gives:

$$\frac{\partial F}{\partial t} = -i \int_0^\infty e^{-it\,(w^2-z^2)} \, dw \tag{6.7}$$

$$= -ie^{itz^2} \int_0^\infty e^{-itw^2} \, dw$$

The integral here equals $\frac{1}{2}\sqrt{\pi}\, e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{t}}$, so we have:

$$\frac{\partial F}{\partial t} = -\tfrac{1}{2}\sqrt{\pi} \; e^{i\frac{\pi}{4}} \frac{e^{itz^2}}{\sqrt{t}} \tag{6.8}$$

Integrating with respect to t now gives:

$$F(t,z) = F(o,z) - \tfrac{1}{2}\sqrt{\pi}\, e^{i\frac{\pi}{4}} \int_0^t \frac{e^{i\tau z^2} d\tau}{\sqrt{\tau}} \tag{6.9}$$

To evaluate F(o,z), we recall that z has a positive imaginary part. The integral (6.6) for t = o then has only one pole in the upper half-plane, and evaluating the residue at that pole gives $F(o,z) = \pi i/2z$. The transformation $\tau z^2 = \frac{\pi}{2} w^2$ then throws (6.9) into the form:

$$F(t,z) = \frac{\pi i}{2z} - \tfrac{1}{2}\pi i\, (1-i)\, \frac{1}{z} \int_0^{\sqrt{\frac{2t}{\pi}}\, z} e^{i\frac{\pi}{2}w^2} \, dw \tag{6.10}$$

Setting t = 1 and inserting into (6.5) gives:

$$J = e^{if(w_o)}\, \pi i e^{-iz^2} \left[1 - (1-i) \int_0^{\sqrt{\frac{2}{\pi}}\, z} e^{i\frac{\pi}{2}w^2} \, dw \right] \tag{6.11}$$

We now wish to compare this to the formula (4.12) to show the superiority of (6.11). It is convenient to remove a factor $2\pi i$ from both (4.12) and (6.11). This is because when we consider the case $\mathrm{Im}\, w_o < \mathrm{Im}\, w_1$ there will be a pole-type contribution to the integral, and it is desirable to make this of unit amplitude.

The integral in (6.11) is in the standard form of Fresnel's integrals(7). We will write u for the upper limit in (6.11), to agree with standard notation. We then have the function:

(6.12)
$$G(u) = \tfrac{1}{2}\, e^{-i \frac{\pi}{2} u^2} \left[1 - (1-i) \left\{ C(u) + iS(u) \right\} \right]$$

where C(u) and S(u) are defined by:

(6.13)
$$C(u) = \int_o^u \cos \frac{\pi}{2} w^2 \, dw$$

$$S(u) = \int_o^u \sin \frac{\pi}{2} w^2 \, dw$$

Upon expressing (4.12) in terms of u, we have also:

(6.14)
$$G_\infty (u) = \frac{1}{\pi \sqrt{2}}\, e^{i \frac{\pi}{4}}\, \frac{1}{u}$$

To justify the notation $G_\infty (u)$, we evaluate G(u) for u large. We then have the approximations correct to order 1/u:

(6.15)
$$C(u) \sim \tfrac{1}{2} + \frac{\sin \frac{\pi}{2} u^2}{\pi u}$$

$$S(u) \sim \tfrac{1}{2} - \frac{\cos \frac{\pi}{2} u^2}{\pi u}$$

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(7) E. Jahnke and F. Emde, "Tables of Functions", Dover 1945, p. 34

We then have for G(u):

$$(6.16)\qquad G(u) \sim \tfrac{1}{2}\, e^{-i\frac{\pi}{2}u^2}\left[1-(1-i)\left(\tfrac{1}{2}+\frac{\sin\frac{\pi}{2}u^2}{\pi u}+\tfrac{1}{2}\, i-\frac{i\cos\frac{\pi}{2}u^2}{\pi u}\right)\right]$$

$$\sim \tfrac{1}{2}\, e^{-i\frac{\pi}{2}u^2}\left[1-\left\{1-i\ (1-i)\ \frac{1}{\pi u}\, e^{i\frac{\pi}{2}u^2}\right\}\right]$$

$$\sim \tfrac{1}{2}\, e^{-i\frac{\pi}{2}u^2}\ i(1-i)\ \frac{1}{\pi u}\, e^{i\frac{\pi}{2}u^2}$$

$$\sim \frac{1}{\pi\sqrt{2}}\, e^{i\frac{\pi}{4}}\ \frac{1}{u}$$

Hence, for u large, G(u) reduces to $G_\infty(u)$, so (6.11) and (4.12) agree when u is large. Now large values of u correspond to the pole at w_1 not being too close to the stationary point at w_0, so we would expect the two formulas to agree. However, (6.11) is valid for all values of u. In particular, for $u = 0$, $G(u) = \frac{1}{2}$. Thus, on the barrier line, the field E has the value $\frac{1}{2}\, h(w_1)\, e^{if(w_1)}$. The plane wave part of the field F due to the pole at a has the value $h(w_1)\, e^{if(w_1)}$, so the total field on the barrier line is just half the field where the plane wave is present.

The analysis for Im $w_0 >$ Im w_1 is very similar. In this case, we have to add a contribution due to the pole, and obtain:

$$(6.17)\qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h(w)}{w-w_1}\, e^{if(w)}\,dw = h(w_1)e^{if(w_1)} + \frac{1}{2\pi i}\int_{\underset{w_0}{\rightarrow}}\frac{h(w)}{w-w_1}\, e^{if(w)}\,dw$$

$$\simeq h(w_1)\, e^{if(w_1)} + \frac{1}{2\pi i}\, h(w_0)\, e^{if(w_0)}\int_{-\infty}^{\infty} e^{\frac{1}{2}if''(w_0)u^2}\ \frac{du}{u+(w_0-w_1)}$$

The integral here is evaluated by the same transformations that led to (6.11).

Calling $u = \frac{1}{\sqrt{\pi}} \sqrt{|f''(w_o)|}\ (w_o - w_1)$, (6.17) becomes:

$$h(w_1)\ e^{if(w_1)} - h(w_o)\ e^{if(w_o)}\ G(u) \tag{6.18}$$

For $u = o$, this has the value $\frac{1}{2}h(w_1)\ e^{if(w_1)}$, and hence the field is continuous upon crossing the barrier line. When u is large, G(u) is small, and hence we are left with the plane wave term $h(w_1)\ e^{if(w_1)}$.

Graphs of the magnitude and phase of G(u) are given in Figures 12 and 13. We see that the magnitude of G(u) decreases steadily as u increases, and the asymptotic form is already indistinguishable from the exact formula when $u = 2$. The phase tends rather slowly to $\pi/4$, differing by a quantity of order $1/u$ and performing oscillations of decreasing magnitude and increasing frequency as u increases.

Let us now apply these general considerations to a particular case. We will treat as a typical example the reflected wave in the neighborhood of the barrier B1 (see Figure 4). We then have $w_o = - k_1 \cos \theta$, $w_1 = - k_1 \cos (2\alpha - \varphi)$. Also $f(w_o) = k_1 r$, $f(w_1) = k_1 r \cos\left[\theta - (2\alpha - \varphi)\right]$, $h(w_o) = \sin^2\alpha / k_1^2 (\cos\theta - \cos\varphi)^2$, $h(w_1) = 1/4k_1^2 \sin^2(\alpha - \varphi)$, $|f''(w_o)| = \frac{r}{k_1 \sin^2\theta}$. We then have for u:

$$u = \frac{1}{\sqrt{\pi}} \frac{1}{\sin\theta} \sqrt{k_1 r}\ \left|\cos\theta - \cos(2\alpha - \varphi)\right| \tag{6.19}$$

Thus, for a fixed angular separation between θ and $2\alpha - \varphi$, u increases as the square root of the distance from the origin. Now the width of the region within which G(u) differs from the asymptotic form $G_\infty(u)$ is from $u = o$ to $u = 2$. Thus for $k_1 r$ large, the angular width of the transition region near the barrier becomes small.

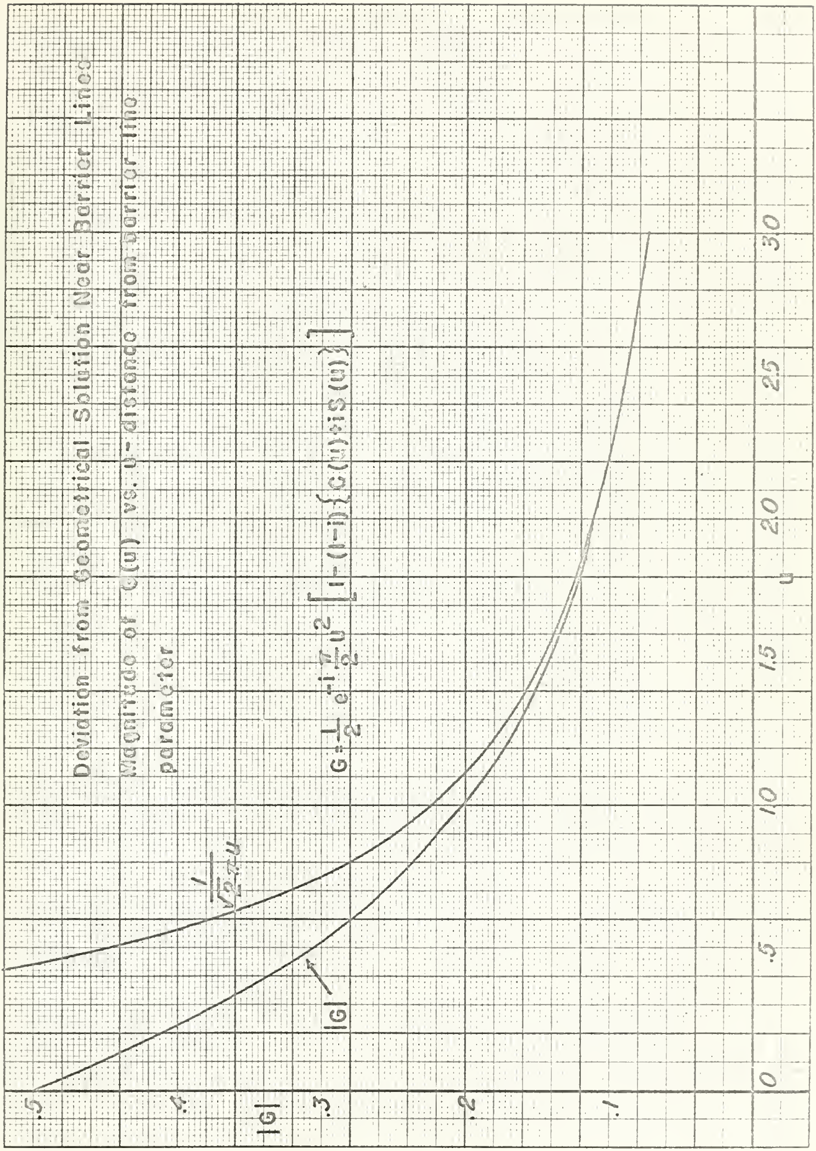
Deviation from Geometrical Solution Near Barrier Lines
Magnitude of G(u) vs. u - distance from barrier line parameter
$G = \frac{1}{2} e^{-i \frac{\pi}{2} u^2} \left[1 - (1-i) \left\{ C(u) + iS(u) \right\} \right]$
$\frac{1}{\sqrt{2}\pi u}$
|G|
|G|
.5
.4
.3
.2
.1
0
.5
1.0
1.5
2.0
2.5
3.0
u

Fig. 12

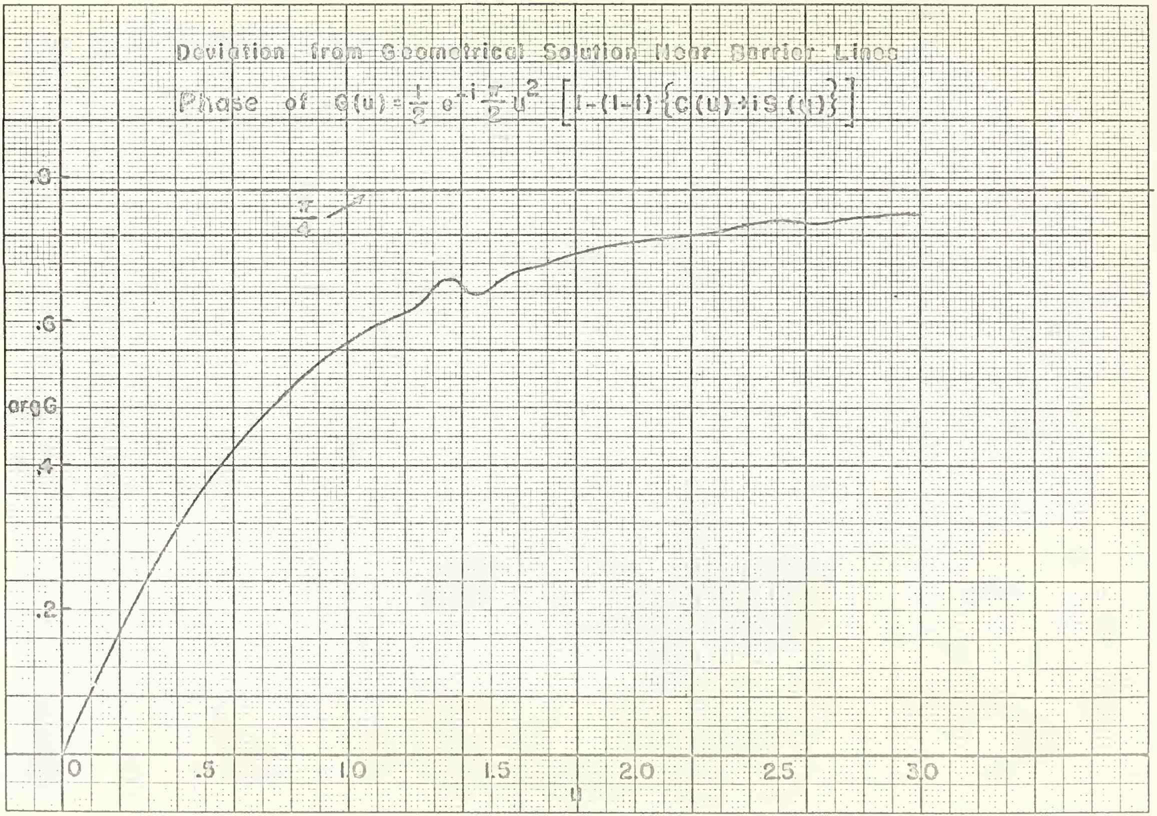
Deviation from Geometrical Solution Near Barrier Lines
Phase of $G(u)=\frac{1}{2}e^{-i\frac{\pi}{2}u^2}\left[1-(1-i)\left\{C(u)+iS(u)\right\}\right]$
.8
.6
arg G
.4
.2
$\frac{\pi}{4}$
0
.5
1.0
1.5
2.0
2.5
3.0
u

Fig. 13

In the transition region we may then approximate u by:

$$u = \frac{1}{\sqrt{\pi}} \sqrt{k_1 r} \; \left| \theta - (2\alpha - \varphi) \right| \tag{6.20}$$

Thus, the total angular width of the transition region thru which E varies from a plane wave plus a cylindrical wave to just a cylindrical wave is given by:

$$\Delta \theta = \frac{4\sqrt{\pi}}{\sqrt{k_1 r}} = \frac{7.09}{\sqrt{k_1 r}} \tag{6.21}$$

since u must vary from 0 to 2 on both sides of the barrier.

This result shows that for $k_1 r$ large, the formula (4.12) as derived by stationary phase is valid everywhere except within a range of angular half-width $\Delta\theta = 2\sqrt{\pi} / \sqrt{k_1 r}$ around the barrier lines. Thus, the disappearance of the plane wave upon crossing the barrier takes place in an angular range whose width decreases steadily as the distance from the origin increases. For $k_1 r$ very large, the transition region is very small, and a geometrical shadow results. The region given by (6.21) thus represents a penumbra region, between "lit" and "unlit" regions.

The curve u = 2 is to good approximation a parabola with the line $\theta = 2\alpha - \varphi$ as axis. Only the part of this parabola for $k_1 r$ large is of significance, since only then can we make the approximation that most of the integral (6.2) comes from the neighborhood of $u = u_o$. This limiting parabola outside of which the plane and cylindrical wave solution is valid has appeared in the theory of diffraction by a reflecting half-plane (8).

To sum up, we see that the formula (6.11) provides a continuous transition for the field E across the barrier line, with a width given by (6.21). On the barrier, the field is reduced to just half its "lit side" value.

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(8) B. Baker and E. Copson, "Huygen's Principle", Oxford 1939, p. 145

7. Application to Propagation Through a Cold Front

The mathematical problem of the dielectric wedge is fairly well realized in practice by the diffraction of plane waves by a "cold front". This consists of a mass of cold air moving into a warmer region. Since the cold air is denser, it forces the warm air to rise over it, and the interface is quite closely a slanting line. Actually, of course, the air in the immediate neighborhood of the interface is quite turbulent.

To isolate the effects, we shall make a number of approximations. First, we will neglect the curvature of the earth, since we are concerned with fairly small regions of the earth's surface. Second, we shall assume the earth to be perfectly conducting, which is well satisfied for most frequencies used in practice. Third, we will neglect the turbulent zone near the interface. Under these simplifying assumptions we may use the solution we have obtained in Sections 2 - 6 to discuss the propagation of waves thru a cold front. The theory is identical for a warm front, since only the direction of motion of the air masses, which we neglect as small compared to the velocity of light, serves to distinguish between the cold and warm fronts.

For an idea of the order of magnitude of the effects involved, we use the well-known formula(9):

$$(n-1) \times 10^6 = \frac{79}{T}\left(p-e-\frac{4800e}{T}\right) \tag{7.1}$$

Here n is the index of refraction, T the temperature in degrees Kelvin, p the pressure in millibars, and e the partial pressure of water vapor in millibars. We also have e = .0016ps, where s is practically equal to

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(9) Burrows and Attwood, "Radio Wave Propagation", Academic Press 1949

the specific humidity. From these we easily derive:

$$dn \times 10^6 = -\frac{dT}{T}(n-1)\times 10^6 + \frac{dp}{p}(n-1)\times 10^6 + \frac{79p}{T}\left(-.0016ds - \frac{7.7ds}{T} + \frac{7.7s\,dT}{T^2}\right) \tag{7.2}$$

Now let us assume that one of the two media is a dry atmosphere at sea level pressure and temperature, so for it p = 1000 mb, T = 300° K, and s = 0. We shall take our figures for the change upon crossing the cold front from a typical weather map, and take dp = 5 mb, dT = 5° K. Both of these are somewhat large. Also, we shall assume the atmosphere is saturated on the wet side, so ds = 1. This also is an extreme figure. Now for these we obtain:

$$\begin{aligned} (n-1)\times 10^6 &= 263 \\ dn \times 10^6 &\cong -.038 \times 263 \end{aligned} \tag{7.3}$$

We have, therefore, a small change in dielectric constant upon crossing the cold front compared to the dielectric constant itself, and the approximations of Sections 2 - 6 are valid.

We shall now consider some practical applications. Thus, consider an airplane traveling thru the cold front, going from region 1 to region 2. How will its received signal vary with position if it flies at constant height? We shall assume that the received signal is proportional to the magnitude of the field strength at the airplane, and will neglect variations of the signal over regions of space comparable to the dimensions of the receiving antenna, which thus will be treated as a point.

As long as the plane is far to the left, and below the barrier line B_2*, the received signal is determined by the incident wave, and thus is proportional to $\sin(k_1 y \sin\varphi)$, which is constant, that is, independent

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*See Fig. 4.

of x, at constant height y. After crossing the barrier line B_2, the field is the sum of the incident field and the reflected wave RL_2. We assume that $k_1 y$ is large, so the diffracted cylindrical wave is negligible compared to the reflected plane wave. From equation (4.6) the intensity of the received signal is proportional to:

$$|E| \sim 2\sin(ky\sin\varphi) + \frac{p}{2k^2}\frac{1}{\sin^2(\alpha+\varphi)}\cos\left\{ky\sin(2\alpha+\varphi) - 2kx\sin(\alpha+\varphi)\sin\alpha\right\} \tag{7.4}$$

Hence, the signal consists of a constant part and a part which oscillates with distance; that is, the received signal is amplitude modulated. Let the velocity of the plane in meters/second be v. Then the frequency of the modulation is:

$$f_2 = f_0 \cdot 2\frac{v}{c}\sin\alpha\sin(\alpha+\varphi) \tag{7.5}$$

where f_0 is the carrier frequency ($f_0 = \omega/2\pi$). If the carrier is in the broadcast band, $f_0 \sim 10^6$ cps, and the velocity is 300 mph, then $2 f_0 v/c \sim 1$ cps and the modulation is inaudible. On the other hand, for microwave communication, $f_0 \sim 3 \times 10^9$ cps, we have $2 f_0 v/c \sim 3$ kc, so the modulation is in the audio region and will be quite noticeable even though the modulation coefficient $p/2k^2\sin^2(\alpha+\varphi)$ is on the order of 4%. This result is a typical example of the inverse Doppler effect(10) (moving receiver, fixed target), and is quite understandable on that basis. It also may of course be viewed as due to interference between the incident wave and the reflected wave RL_2.

When the plane crosses the barrier line B_1, the wave RL_1 is present, and this produces further modulation of the carrier. In addition to the frequency f_2, the received signal also contains the frequency f_1, where:

$$f_1 = f_0 \cdot 2\frac{v}{c}\sin\alpha\sin(\alpha-\varphi) \tag{7.6}$$

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(10) E. J. Barlow, Proc. I.R.E., 37, 340 (1949)

In case φ is a small angle, so the frequencies are close together, the modulation will display beats, where the beat frequency is given by:

(7.7)
$$f_B = f_o \, 2 \, \frac{v}{c} \sin 2\alpha \sin \varphi$$

and of course f_B is the difference between f_1 and f_2.

As the plane continues, the same general behavior ensues. The wave displays modulation at rather low frequencies between the interface and the barrier line B_2. This modulation arises from interference between the two refracted waves, which are of the same order of magnitude. Thus modulation is maintained throughout region 2. It may be thought of as arising from the difference in path length of the directly received wave and the wave reflected from the ground, which path difference is proportional to the total path length and thus acts as a modulation. The frequency of this modulation is:

(7.8)
$$f = \tfrac{1}{2} f_o \frac{v}{c} \frac{k_2^2 - k_1^2}{k_1^2} \frac{\sin\alpha \cos\alpha \sin\varphi}{\sin(\alpha - \varphi)\sin(\alpha + \varphi)}$$

Because of the factor $k_2^2 - k_1^2$, the effective frequency may be quite low.

In addition to producing modulation, the effect of refraction on the transmitted wave is such as to cause it to be deviated. Hence, a plane following the beam to a landing may not actually be homing. If the plane is flying toward the source, the vertical deviation of the beam is in such a sense that the plane will fly upward instead of horizontally. Also, if the beam is not normally incident on the interface, there will be a horizontal deviation, which will cause the plane to follow a course toward the apparent source rather than the true source. It should be noted that all these effects may be masked by the effect of the turbulent zone near the interface.

This completes the discussion of the spatial behavior of the signal. We shall now recapitulate the results. Using Maxwell's equations and suitable approximate methods of calculation, we have obtained the fields yielded by geometrical optics and have found the modifications on them produced by diffraction. These results have been expressed as combinations of waves traveling in the proper directions, and the amplitudes of these waves have been found to first order in the difference in the propagation constants of the media. Under the conditions of incidence which have been treated here, no new waves will appear in the higher order terms in the expansion, nor will the directions of propagation be modified, although the amplitudes of course will be affected. When the incidence is from region 2, and the angle is small, new waves will appear in higher approximations, as indicated by Figure 6, but their amplitudes will be small. We also have discussed the transition of the field across the barrier lines.

The alternative polarization (H horizontal), may be treated by the same methods. The details of the analysis are slightly more complicated, because the boundary conditions at the interface are different, but the results should show the same general character. The same approach might also be used to treat such questions as the wave theory for a prism and the diffraction by bodies of polygonal cross-section.

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